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Eigenvalue bounds for independent sets

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Abstract

We derive bounds on the size of an independent set based on eigenvalues. This generalizes a result due to Delsarte and Hoffman. We use this to obtain new bounds on the independence number of the Erdős–Rényi graphs. We investigate further properties of our bounds, and show how our results on the Erdős–Rényi graphs can be extended to other polarity graphs.

Keywords: Independent sets; Delsarte–Hoffman bound; Polarity graph; Loops

1. Introduction

Let $\mathbb{F}$ be a finite field of order q and let V be a 3-dimensional vector space over $\mathbb{F}$. The 1-dimensional subspaces of V are the points of the projective plane $PG(2,q)$, and the 2-dimensional subspaces are the lines. It follows that each point can be represented by a non-zero vector, namely any vector that spans the corresponding 1-dimensional subspace. Two points a and b, represented by vectors x and y respectively, are *orthogonal* if $x^T y = 0$. The Erdős–Rényi graph $ER(q)$ is the graph with the points of $PG(2,q)$ as its vertices, where two vertices are adjacent if and only if they are orthogonal.

The graph $ER(q)$ has q^2+q+1 vertices and each vertex has exactly $q+1$ neighbors. This is not a simple graph: by standard results in finite geometry, there are exactly $q+1$ vertices that are adjacent to themselves. Thus our graph has $q+1$ loops. The Erdős–Rényi graphs are of interest because they do not contain any 4-cycles, but nonetheless they have a large number of edges; this is the motivation for [3,6]. For further work on these graphs, see [7,8,13,16].

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Our object in this paper is to derive good bounds on the size of an independent set in $ER(q)$. For our purposes an independent set is a subset of the vertices such that no two distinct vertices are adjacent. Thus an independent set may contain vertices with loops. For a graph X, let $\alpha(X)$ denote the maximum number of vertices in an independent set in X. There is a standard bound for $\alpha(X)$ in terms of the eigenvalues of X, due to Delsarte and Hoffman (see [5, Section 3.3] or [4, p. 115]; alternatively [12] for more recent work). However this bound only applies to regular graphs with no loops, and consequently our first task in this paper is to derive an extension of it. With this in hand we are able to derive new (and better) bounds on the size of independent sets in $ER(q)$.

We conclude the paper by describing some more general classes of graphs to which our new bound can be applied. These are obtained as follows. Suppose Y is a connected k-regular bipartite graph on $2v$ vertices, and suppose there is an automorphism θ of Y with order two that swaps the two color classes of Y. The *quotient graph* Y/θ is a graph with the v orbits of θ as its vertices, and with w_{ij} arcs from orbit i to orbit j, where w_{ij} is the number of edges in Y from a vertex in orbit i to the vertices of orbit j. Any orbit of θ that contains two adjacent vertices gives rise to a vertex in Y/θ with a loop. It is not hard to show that $w_{ji} = w_{ij}$ and that, if Y has no 4-cycles, then Y/θ does not have any multiple edges. The graph $ER(q)$ can be constructed in this way from the incidence graph of the projective plane $PG(2,q)$; the vertices of the incidence graph are the points and lines of $PG(2,q)$ and a point is adjacent to a line in the graph if it is incident with the line in the geometry. The map that sends the point represented by the non-zero vector x to the line consisting of the points represented by the non-zero vectors y such that $x^T y = 0$ gives rise to an automorphism θ of order two that swaps points and lines.

2. General framework

We concern ourselves with bounding the size of an independent set in a graph. We will permit loops on vertices, but we will allow these vertices to be included in an independent set; in other words, we define an independent set to be a set of vertices of which no two distinct members are adjacent. Allowing loops is the more general option. If we later wish to exclude them we may delete the looped vertices.

We will need a little linear algebra. Recall that a symmetric matrix B is *positive semidefinite* if all of its eigenvalues are non-negative; equivalently, if $x^T Bx \geqslant 0$ for all vectors x. We write $B \succeq 0$. If B is positive semi-definite, then $x^T Bx = 0$ if and only if $Bx = 0$.

Let X be a graph with vertex set V, $|V| = n$, possibly containing loops, and let A be its adjacency matrix. Let $T = \operatorname{diag}(t_1, \dots, t_n)$ be a diagonal matrix such that $T + A \succeq 0$. Also, let d_i be the degree of vertex i, with loops counted once each. Consider an independent set S of size s, and let s_1 be the number of loops on vertices in S. Let z be the characteristic vector of S. Then we have

$$\left(z - \frac{s}{n}\mathbf{1}\right)^T (T + A)\left(z - \frac{s}{n}\mathbf{1}\right) \geqslant 0.$$

Expanding this we obtain the following result.

Lemma 2.1. *Let X be a graph with vertex set V, $|V| = n$ and vertex degrees $d_1, \dots, d_n$. Let A be its adjacency matrix, and let $T = \operatorname{diag}(t_1, \dots, t_n)$ be such that $T + A \succeq 0$. If S is an independent set of size s containing s_1 loops, then*:

$$\frac{s^2}{n^2}\sum_{i\in V}(t_i + d_i) - 2\frac{s}{n}\sum_{i\in S}(t_i + d_i) + \sum_{i\in S} t_i \geqslant -s_1.$$

This gives a bound on s. However, it is difficult to apply in general, partly because the sums depend not only on s but on S. Furthermore, the bound obtained will depend on the choice of T. We do not know how to choose T optimally (or even if there is a single optimal choice for all graphs). In the present paper, we consider specific choices for T.

It will be useful to define the following parameters of a set S:

$$\bar{d}_S = \frac{1}{s}\sum_{i\in S} d_i,$$

$$k_S = 2\bar{d}_S - \frac{1}{n}\sum_{i\in V} d_i.$$

Note that for k-regular graphs, $\bar{d}_S = k_S = k$. It will be seen that these two parameters behave, in some circumstances, as analogues to the degree of a regular graph.

3. Bounds

We consider two particular choices for T, producing bounds which we can regard as coming from, respectively, the adjacency matrix and the Laplacian matrix of the graph.

3.1. Adjacency matrix

If we let τ be the least eigenvalue of A, then we may set $T = -\tau I$ giving $T + A = A - \tau I \succeq 0$. If X is regular and loopless, then simplification of Lemma 2.1 gives the Delsarte–Hoffman bound, which we state as the following corollary.

Corollary 3.1. *Let X be a k-regular graph with no loops, and τ the least eigenvalue of its adjacency matrix. For any independent set S of size s, we have*:

$$s \leqslant n\frac{-\tau}{k-\tau}.$$

It turns out that X need be neither regular nor loopless.

Corollary 3.2. *Let X be a graph with no loops, and τ the least eigenvalue of its adjacency matrix. For any independent set S of size s, we have*:

$$s \leqslant n\frac{-\tau}{k_S-\tau}.$$

Note that k_S plays an analogous role to that of the degree in Corollary 3.1. However, k_S can be zero or even negative: the bound is then useless. To be precise, one should say that Corollary 3.2

does not bound the size of an independent set, but provides a family of bounds, one for each value of k_S (or equivalently, one bound for each value of $\bar{d}_S$). Any lower bound on $\bar{d}_S$ (such as the minimum degree) can be used to make Corollary 3.2 into a true bound on s.

If X has loops, then Lemma 2.1 is a nontrivial quadratic. The bounds we get are slightly messier, but again, in the non-regular case, k_S plays a role analogous to that of the degree in the regular case.

Corollary 3.3. *Let X be a k-regular graph with loops, and τ the least eigenvalue of its adjacency matrix. For any independent set S of size s containing s_1 loops, we have*:

$$s \leqslant n\frac{-\tau + \sqrt{\tau^2 + 4s_1\frac{k-\tau}{n}}}{2(k-\tau)}.$$

Corollary 3.4. *Let X be a graph with loops, and τ the least eigenvalue of its adjacency matrix. For any independent set S of size s containing s_1 loops, we have*:

$$s \leqslant n\frac{-\tau + \sqrt{\tau^2 + 4s_1\frac{k_S-\tau}{n}}}{2(k_S-\tau)}.$$

3.2. Laplacian matrix

For a graph X with adjacency matrix A and diagonal matrix of degrees D, recall that $L = D - A$ is the Laplacian matrix of X. We always have $L \succeq 0$, and 0 is an eigenvalue of multiplicity equal to the number of components of X. The greatest eigenvalue of L is at most twice the maximum degree; it is also bounded by the number of vertices (see for instance [1]). If X is regular then $L = kI - A$ and the eigenvalues of L determine the eigenvalues of A and vice versa. Accordingly we expect to recover previous bounds for regular graphs and hope to obtain new ones in the non-regular case. Note that graphs that differ only by the presence or absence of loops have the same Laplacian matrix. Thus, without loss of generality, we can assume that the graph has no loops and set $s_1 = 0$.

If we let μ be the greatest eigenvalue of L, then we may set $T = \mu I - D$ giving $T + A = \mu I - L \succeq 0$. If the graph is regular, then we recover Corollary 3.1, as expected. If it is not regular, then we obtain the following bound.

Corollary 3.5. *Let X he a loopless graph, and μ, the greatest eigenvalue of its Laplacian matrix. For any independent set S of size s, we have*:

$$s \leqslant n\frac{\mu - \bar{d}_S}{\mu}.$$

Note that $\bar{d}_S$ plays an analogous role to that of the degree in Corollary 3.1. We could have stated Corollary 3.5 for graphs with loops, but it is more convenient to leave them out.

This result generalizes Corollary 3.1, but it also generalizes Corollary 3.3. Let X be a k-regular graph with loops and let Y be X with the loop-edges deleted. The Laplacian eigenvalues of X and Y are identical, so

$$\mu(X) = \mu(Y) = k - \tau(Y).$$

Also,

$$\bar{d}_S(X) = k - \frac{s_1}{s}.$$

Substituting into Corollary 3.5 yields Corollary 3.3.

We can weaken Corollary 3.5 slightly to a more usable form, by noting that $\bar{d}_S \geqslant \delta$.

Corollary 3.6. *Let X be any graph with minimum degree δ, and μ the greatest eigenvalue of its Laplacian matrix. For any independent set S of size s, we have*:

$$s \leqslant n\frac{\mu - \delta}{\mu}.$$

Corollary 3.5 was obtained independently by Lu, Liu and Tian [11]. Corollaries 3.2 and 3.5 were also obtained using interlacing by Bollobás and Nikiforov [2].

4. Equality

If Lemma 2.1 holds with equality then it follows that

$$(T + A)\left(z - \frac{s}{n}\mathbf{1}\right) = 0,$$

and we have an eigenvector for $T + A$. Unpacking this equality gives a proof of the following.

Lemma 4.1. *Let X be a graph with vertex degrees $d_1, \dots, d_n$ and adjacency matrix A. Let $T = \operatorname{diag}(t_1, \dots, t_n)$ be such that $T + A \succeq 0$. If S is a set of s vertices with no two distinct vertices adjacent such that Lemma* 2.1 *holds with equality, then*:

(a) *Each vertex i in S has degree $d_i = t_i(\frac{n}{s} - 1)$.*
(b) *Each vertex i not in S has $(d_i + t_i)\frac{s}{n}$ neighbors in S.*

4.1. Adjacency matrix

If S is an independent set in a k-regular graph such that Corollary 3.1 holds with equality, then we have an eigenvector for A:

$$A\left(z - \frac{s}{n}\mathbf{1}\right) = \tau\left(z - \frac{s}{n}\mathbf{1}\right).$$

Since the vector $\mathbf{1}$ spans the k-eigenspace, z lies in the sum of the greatest eigenspace and the least eigenspace.

Using Lemma 4.1, we see that the bipartite subgraph induced by the partition $\{S, V(X)\backslash S\}$ is semi-regular: vertices in S have $-\tau(\frac{n}{s} - 1)$ neighbors not in S, and vertices not in S have $(k - \tau)\frac{s}{n}$ neighbors in S. But vertices in S have k neighbors not in S, since the graph is k-regular. It follows that vertices not in S have $-\tau$ neighbors in S.

An *equitable partition* is a partition of the vertex set such that for any vertex a and cell C, the number of neighbors of a in C depends only on which cell a is contained in. Note that in the quotient graph of Section 1, the orbits of θ form an equitable partition. In the previous paragraph, the partition $\{S, V(X)\backslash S\}$ is *equitable.*

We summarize our findings in the following, due to Delsarte and Hoffman (unpublished).

Theorem 4.2. *Let X be a k-regular graph with no loops, and τ the least eigenvalue of its adjacency matrix. For any independent set S of size s and characteristic vector z, we have*:

$$s \leqslant n\frac{-\tau}{k-\tau}.$$

Furthermore, if this bound holds with equality, then:

(a) *z is a linear combination of a k-eigenvector and a τ-eigenvector.*
(b) *The bipartite subgraph induced by the partition $\{S, V(X)\backslash S\}$ is semi-regular.*
(c) *The partition $\{S, V(X)\backslash S\}$ is equitable.*

The k-eigenvectors of a regular graph are exactly the constant vectors. For non-regular graphs, the bounds based on the adjacency matrix seem less useful. Specifically, if equality holds in Corollary 3.2, Corollary 3.3, or Corollary 3.4, then we still have an eigenvector for A, but as $\mathbf{1}$ is no longer an eigenvector, the conditions of Lemma 4.1 are not as useful.

4.2. Laplacian matrix

If S is an independent set in a graph such that Corollary 3.5 holds with equality, then we have an eigenvector for L:

$$L\left(z-\frac{s}{n}\mathbf{1}\right)=\mu\left(z-\frac{s}{n}\mathbf{1}\right).$$

Since the vector $\mathbf{1}$ spans the 0-eigenspace, we have that z lies in the sum of the least eigenspace and the greatest eigenspace. Again, from Lemma 4.1, we see that the bipartite subgraph induced by the partition $\{S, V(X)\backslash S\}$ is semi-regular: vertices in S have $\mu(1-\frac{s}{n})$ neighbors outside of S and vertices outside of S have $\mu\frac{s}{n}$ neighbors in S.

This does not quite say that $\{S, V(X)\backslash S\}$ is an equitable partition: the missing condition needed is that every vertex in $V(X)\backslash S$ would have a constant number of neighbors in $V(X)\backslash S$. If this condition were to hold then all vertices not in S would have the same degree.

We summarize our findings as follows.

Theorem 4.3. *Let X be a graph with no loops, and μ the greatest eigenvalue of its Laplacian matrix. For any independent set S of size s and characteristic vector z, we have*:

$$s \leqslant n\frac{\mu-\bar{d}_S}{\mu}.$$

Furthermore, if this bound holds with equality, then:

(a) *z is a linear combination of a 0-eigenvector and a μ-eigenvector.*
(b) *The bipartite subgraph induced by the partition $\{S, V(X)\backslash S\}$ is semi-regular.*

The 0-eigenvectors of the Laplacian are exactly the constant vectors; compare this to Theorem 4.2, where the k-eigenvectors were the constant vectors, but only because the graph was regular. Note furthermore that in both cases, when equality holds, z is a linear combination of eigenvectors belonging to the greatest and least eigenvalues. Based partly on the analogy between Theorem 4.2 and Theorem 4.3, and the fact that Theorem 4.2 is actually a special case of Theorem 4.3, it seems that the Laplacian matrix formulation is the natural generalization to non-regular graphs.

Assume that equality holds in Theorem 4.3 and $\gcd(s,n)=1$. The number of vertices in S adjacent to a vertex not in S, $\mu\frac{s}{n}$, must be an integer, and so $n \mid \mu$. As $0<\mu\leqslant n$, it follows that $n=\mu$, and the bipartite subgraph induced by the partition $\{S, V(X)\backslash S\}$ is complete bipartite. Of course the same conclusion follows if equality holds in Theorem 4.2, since it is a special case of Theorem 4.3.

5. Comparing bounds

The bounds of Corollary 3.2 and Corollary 3.5 are not directly comparable: we show this using examples. We start with a simple family of graphs where Corollary 3.5 is better, then a family where neither bound is uniformly better.

Consider the graphs $K_{a,b}$ where $a<b$. Clearly the only maximum independent set is the set of vertices of degree a. So we have

$$\bar{d}_S = a, \qquad k_S = \frac{2a^2}{a+b}.$$

Also, the least eigenvalue of the adjacency matrix is $-\sqrt{ab}$ and the greatest eigenvalue of the Laplacian matrix is $a+b$. Applying Corollary 3.2 we get

$$s \leqslant \frac{(a+b)^2\sqrt{ab}}{(a+b)\sqrt{ab}+2a^2}. \tag{1}$$

However applying Corollary 3.5 we get exactly the size of the maximum independent set:

$$s \leqslant b.$$

Thus we conclude that the bounds based on the adjacency and Laplacian matrices are not equal (for instance, for $K_{4,23}$(1) gives $s \leqslant 24$). Furthermore, not only is Corollary 3.5 tight, but so is Corollary 3.6. The latter bound is in terms of the graph only, whereas the former depends on $\bar{d}_S$ and so retains an implicit dependence on the structure of S.

Now consider the graphs X_m, $m>1$, constructed as follows. Let G_m be a copy of $\overline{K_m}$, and H_m be a copy of C_{2m+1}. Then X_m consists of the disjoint union of G_m and H_m, together with edges from every vertex of G_m to every vertex of H_m.

Clearly, the maximum independent sets are of size $s=m$ and there are two types: the vertices of G_m and a maximum independent set in H_m. The first type has $\bar{d}_S = 2m+1$ and the second has $\bar{d}_S = m+2$. Owing to the block structure of the adjacency and Laplacian matrices, the eigenvectors can be determined. We find that $\tau = 1-\sqrt{2m^2+m+1}$ and $\mu = 3m+1$.

Using this information we can compute the two bounds. For convenience, let α_A and α_L be the values of the bounds in Corollary 3.2 and Corollary 3.5, respectively. If S is the vertex set of G_m; we determine that the Laplacian bound is tight while the adjacency bound is not:

$$|S| = m = \alpha_L < \alpha_A.$$

Now let S be a maximum independent set in H_m. We find that for small m, the adjacency bound is better, and for large m the Laplacian bound is better. Neither bound is ever tight, except that the adjacency bound for $m=2$ is correct when rounded:

$$|S| = m < \alpha_A < \alpha_L = 2m-1 \quad \text{for } 2 \leqslant m \leqslant 24,$$
$$|S| = m < \alpha_L = 2m-1 < \alpha_A \quad \text{for } 25 \leqslant m.$$

5.1. Another eigenvalue bound

We now compare the Delsarte–Hoffman bound of Corollary 3.1 with another eigenvalue bound of Sarnak.

Let λ be the maximum of the second largest eigenvalue and the absolute value of the least eigenvalue of the adjacency matrix. Sarnak [15] has shown the following bound for an independent set S in a k-regular graph.

Lemma 5.1.

$$|S| \leqslant n\frac{\lambda}{k}.$$

It turns out that this is strictly weaker than Corollary 3.1, as we now show. For reference, we give an outline of the proof in [15].

Proof. If $x \perp \mathbf{1}$ then $\|Ax\|^2 \leqslant \|\lambda x\|^2$. We choose

$$x_i = \begin{cases} n-s, & i \in S, \\ -s, & i \notin S \end{cases}$$

and compute the norms as follows:

$$\begin{aligned} \|\lambda x\|^2 &= \lambda^2\big(s(n-s)^2 + (n-s)s^2\big) = \lambda^2 ns(n-s), \\ \|Ax\|^2 &= \sum_{i\in S}\big((Ax)_i\big)^2 + \sum_{i\notin S}\big((Ax)_i\big)^2 \\ &\geqslant \sum_{i\in S}\big((Ax)_i\big)^2 \\ &= s^3k^2. \end{aligned} \tag{2}$$

Substituting into $\|Ax\|^2 \leqslant \|\lambda x\|^2$, we find that

$$s \leqslant \sqrt{n(n-s)}\frac{\lambda}{k}$$

which implies

$$s \leqslant n\frac{\lambda}{k}. \qquad \square \tag{3}$$

Note that at (2) we are neglecting some positive terms, and at (3), we are using $n - s \leqslant n$. Assuming that $0 < s < n$, either of these is sufficient to guarantee that the inequality in Lemma 5.1 is strict. Based on these observations, we can improve this proof. If we write the adjacency matrix in the form

$$A = \begin{pmatrix} 0 & B \\ B^T & C \end{pmatrix},$$

then we can compute the missing contributions as follows:

$$
\begin{aligned}
\sum_{i \notin S} \big((A_X)_i\big)^2 &= \sum_i \big((n-s)\big(B^T\mathbf{1}\big)_i + (-s)(C\mathbf{1})_i\big)^2 \\
&\geqslant \frac{1}{n-s}\Big(\sum_i \big((n-s)\big(B^T\mathbf{1}\big)_i + (-s)(C\mathbf{1})_i\big)\Big)^2 \\
&= \frac{1}{n-s}\big((n-s)(sk) + (-s)\big((n-2s)k\big)\big)^2 \\
&= \frac{s^4k^2}{n-s}.
\end{aligned} \tag{4}
$$

This gives that

$$
\|Ax\| \geqslant \frac{s^3k^2n}{n-s},
$$

and then using $\|Ax\|^2 \leqslant \|\lambda x\|^2$ and rearranging gives exactly

$$
s \leqslant \frac{n}{1+\frac{k}{\lambda}}. \tag{5}
$$

This bound is strictly better than Lemma 5.1, but, as $\lambda \geqslant -\tau$, it is no stronger than Corollary 3.1. In the context of the present paper, this strengthening is not a surprise: the vector x in the proof of Lemma 5.1 is just a multiple of $z - \frac{s}{n}\mathbf{1}$. Note also that if (5) holds with equality then so does (4): in the extremal case we have not neglected any terms.

6. A generalization

In developing the bounds of Section 3, we were motivated by a need to have tools we could apply to the graphs $ER(q)$. It seems that our approach to the Delsarte–Hoffman bound can be pushed further. As one example consider the following.

Theorem 6.1. *Let X be any graph on n vertices, and S and independent set of size s. Let B be a symmetric square matrix indexed by the vertices of X such that*

(a) $B \succeq 0$.
(b) $B_{ij} \leqslant 0$ *whenever* $i \nsim j$ *and* $i \neq j$.
(c) B *has constant row sum* r.
(d) B *has constant diagonal* t.

Then

$$
s \leqslant n\frac{t}{r}. \tag{6}
$$

The proof is analogous to our previous bounds on independent sets. Since $B \succeq 0$ we have

$$
\Big(z - \frac{s}{n}\mathbf{1}\Big)^T B\Big(z - \frac{s}{n}\mathbf{1}\Big) \geqslant 0.
$$

Unpacking this inequality gives the required bound on s. Furthermore, if equality holds in (6), then $z^TBz = ts$, i.e., $B_{ij} = 0$ for any two distinct $i, j \in S$. It follows that equality in (6) implies that for any $i \in S$, we have $\sum_{j \in S} B_{ij} = t$ and $\sum_{j \notin S} B_{ij} = r - t$. Equality in (6) also gives an eigenvector condition:

$$B\left(z - \frac{s}{n}\mathbf{1}\right) = 0,$$

from which it follows that for any $i \notin S$, we have $\sum_{j \in S} B_{ij} = t$ and $\sum_{j \notin S} B_{ij} = r - t$. Hence $\{S, V \backslash S\}$ is a generalization of an equitable partition for weighted graphs.

Not surprisingly if $B = A - \tau I$, where A is the adjacency matrix of X and τ is its least eigenvalue, then we recover Theorem 4.2. The equality conditions just given then reduce to saying that $\{S, V(X) \backslash S\}$ is an equitable partition.

Theorem 6.1 is relevant for the following reason. Let $\mathcal{A} = \{A_0, \ldots, A_d\}$ be an association scheme and let $R \subseteq \{1, \ldots, d\}$ (see [9, Chapter 12] for background and notation). An R-coclique is a set of vertices such that no two of them are i-related for $i \in R$. In other words, it is an independent set in the graph formed by the union of the classes of R.

We denote the trace of a matrix N by $\mathrm{tr}(N)$, and the sum of all of its entries by $\mathrm{sum}(N)$. The span of $\mathcal{A}$ is denoted by $\langle A \rangle$. The following result is shown in [10].

Theorem 6.2. *Let $\mathcal{A}$ be an association scheme with d classes, let $R \subseteq \{1, \ldots, d\}$ and let S be an R-coclique in $\mathcal{A}$. Then*

$$|S| \leqslant \min\left\{ v\frac{\mathrm{tr}(N)}{\mathrm{sum}(N)} \;\middle|\; N \in \langle A \rangle, N \succeq 0, N \circ A_i \leqslant 0 \textit{ for } i \notin R \cup \{0\} \right\}.$$

Let X be the graph formed by the union of the classes of R. Then the matrix N satisfies the criteria of Theorem 6.1 with $tv = \mathrm{tr}(N)$ and $rv = \mathrm{sum}(N)$. So Theorem 6.1 is a generalization of Theorem 6.2.

On the other hand, we may write N in terms of the matrix idempotents and Schur idempotents as

$$N = \sum_i a_i A_i = \sum_j b_j E_j.$$

Thus $\mathrm{tr}(N) = va_0$ and $\mathrm{sum}(N) = vb_0$. Then Theorem 6.2 says that $|S|$ is bounded above by the following linear program:

$$\min\left\{ va_0 \mid b_j \geqslant 0, b_0 = 1, a_i \leqslant 0 \text{ for } i \notin R \cup \{0\}\right\}.$$

This is equal to Delsarte's LP bound on an $\bar{R}$-coclique [5]. In other words, Theorem 6.1 is a generalization of Delsarte's LP bound to general graphs.

7. Applications to finite geometry

We now turn our attention to the motivation of our present work: applying the tools of Section 3 to derive new bounds on the independence number of the Erdős–Rényi graphs. Our technique can be applied to more general polarity graphs.

7.1. Quotients

Let Y be a k-regular graph, and let θ be an automorphism of Y of order 2. We then let $X = Y/\theta$ be the corresponding quotient graph. Recall that X is the graph whose vertices are the orbits of θ, with w_{ij} arcs from orbit i to orbit j, where w_{ij} is the number of edges in Y from a vertex of orbit i to the vertices of orbit j.

Note that in general X will be a weighted digraph, possibly with loops. It will be a graph (i.e., $w_{ij} = w_{ji}$) if and only if there is no edge between an orbit of size one and an orbit of size two. It will have no multiple edges if and only if the edges joining two cells of size two never form a complete bipartite graph. It will have no loops if and only if θ never interchanges adjacent vertices.

The eigenvectors of X correspond to the eigenvectors of Y that are constant on each orbit of θ. In particular, the eigenvalues of X are exactly the eigenvalues of Y corresponding to eigenvectors that are constant on each cell (see [9, Chapter 5] for details). Thus if we know the eigenvectors of $A(Y)$, we know the least eigenvalue of $A = A(X)$, and we may apply Corollary 3.3.

7.2. *Erdős–Rényi graphs*

As a specific example, let Y be the incidence graph of $PG(2,q)$. This is a bipartite graph, with points and lines forming the two color classes. Let θ be the map that sends a point to the line with the same coordinates, and vice versa. Then every orbit of θ has size two and Y/θ is an undirected graph with no multiple edges. It does however have $q+1$ vertices with loops. The graph that results from removing the loops is known as the *Erdős–Rényi* graph of order q. For convenience, we will leave the loops in. The graph X has q^2+q+1 vertices, degree $q+1$, and eigenvalues $q+1, \pm\sqrt{q}$. Let S be an independent set of size s containing s_1 loops in X. We use our results from Section 3 to bound s.

We can apply Corollary 3.1 directly; this will bound the size of an independent set containing no loops (i.e., an independent set that contains no absolute vertices). Our set S could be at most $q+1$ greater than this, yielding the following bound:

$$s \leqslant n\frac{-\tau}{k-\tau} + q + 1 = \frac{(q^2+q+1)\sqrt{q}}{q+\sqrt{q}+1} + q + 1. \tag{7}$$

We can of course use Lemma 5.1 instead of Corollary 3.1; this gives a manifestly weaker bound on s.

A better approach is to use Corollary 3.3. In order to obtain a bound independent of s_1, we set $s_1 = q+1$, to get the following:

$$s \leqslant \frac{\sqrt{q} + \sqrt{q + 4(q+1)\frac{q+\sqrt{q}+1}{q^2+q+1}}}{2\frac{q+\sqrt{q}+1}{q^2+q+1}}. \tag{8}$$

As noted in Section 3.2, this is equivalent to using Corollary 3.5 (or more precisely, Corollary 3.6). Some tedious algebra shows that (8) is strictly better than (7).

Another approach would be to consider the graph X_0, obtained by deleting the absolute vertices from X. Godsil and Royle have computed the characteristic polynomial of the graph X_0 to be

$$(\lambda - q)\lambda(\lambda+1)^q(\lambda^2 - q)^{(q^2-q-2)/2}.$$

So the least eigenvalue of X_0 is $-\sqrt{q}$. Let S_0 be an independent set of average degree $\bar{d}_{S_0}$ in X_0. Trivially, we see that

$$\bar{d}_{S_0} \geqslant q-1,$$

and so we can apply Corollary 3.2 to X_0 and add $q+1$ to obtain the following bound:

$$s \leqslant \frac{q^2\sqrt{q}}{q-2+\frac{1}{q}+\sqrt{q}} + q + 1. \tag{9}$$

On the other hand, we may assume that S_0 is contained in an independent set S of X. Thus S consists of S_0 together with, say, s_1 absolute vertices. Each absolute vertex in X is adjacent with at most q vertices of S_0, so there are at most $q(q+1-s_1)$ edges between S_0 and the set of absolute vertices of X. It follows that

$$\bar{d}_{S_0} \geqslant q+1-\frac{q(q+1-s_1)}{s-s_1}.$$

Substituting this into Corollary 3.2 and adding s_1, we get a bound in terms of s_1. For $q \geqslant 5$, this is a decreasing function of s_1, so we set $s_1 = 0$ to obtain the following:

$$s \leqslant \frac{q^2\sqrt{q}+2q(q+1)}{q+2+\frac{1}{q}+\sqrt{q}}. \tag{10}$$

For $5 \leqslant q \leqslant 23$, (10) is better than (9), but for $q \geqslant 25$, the reverse is true. Neither bound is as good as (8).

Yet another approach is to delete the loop-edges from X and apply Corollary 3.2. (This is the more usual Erdős–Rényi graph.) Godsil and Royle have computed its characteristic polynomial to be

$$\left(\lambda^3 - q\lambda^2 - 2q\lambda + q^2 + q\right)\left(\lambda^2+\lambda+1-q\right)^q\left(\lambda^2-q\right)^{(q^2-q-2)/2}.$$

The least eigenvalue is a root of the cubic factor. We can approximate it using Newton's method. It is less than $-\sqrt{q}$, it is the only eigenvalue that is less than $-\sqrt{q}$, and the cubic factor is concave down for $\lambda \leqslant -\sqrt{q}$. So iterating Newton's method starting with $-\sqrt{q}$ will always give a lower bound on the least eigenvalue, which means we will be overestimating our upper bound on the size of an independent set. In fact, since we only care about the integer part of the final answer, it seems that two iterations is sufficient.

Let S be an independent set of size s containing s_1 absolute vertices (here these no longer have loops). It is straightforward to compute $\bar{d}_S$:

$$\bar{d}_S = q+1-\frac{s_1}{s}.$$

Letting w be an approximation to the least eigenvalue and applying Corollary 3.2 we get

$$s \leqslant \frac{(q^2+q+1)(-w)+2(q+1)}{q+1-w+\frac{q+1}{q^2+q+1}}. \tag{11}$$

We close this section with a brief table summarizing the numerical values of the bounds we have derived (Table 1). We also include exact values for the size of a maximum independent set; these are from Williford [16, Section 4.3]. It is partly the difference between his values and the bound of (7) that motivated our work. The best bound we know of is (8).

Table 1

q	$\alpha(ER(q))$	(8)	(11)	(7)	(9)	(10)
3	5	5.56	5.63	7.92	9.09	6.21
5	10	10.56	10.82	14.42	16.28	12.28
7	15	16.73	17.27	22.16	24.65	20.50
9	22	23.93	24.87	31	34.03	29.98
11	29	32.05	33.40	40.79	44.34	40.55
13	38	41.03	42.88	51.48	55.49	52.08

7.3. Polarity graphs

Much of the work in the previous section can be applied more generally.

Let Y be the incidence graph of $PG(2,q)$, let θ be an automorphism of Y of order two that swaps points and lines of $PG(2,q)$, and let $X = Y/\theta$. The vertices in X with loops are the absolute vertices; denote their number by a. Let N be the adjacency matrix of X. Then N is an incidence matrix of $PG(2,q)$ (the image under θ of the point corresponding to the ith row of N is the line corresponding to the ith column of N), and the adjacency matrix of Y is

$$\begin{pmatrix} 0 & N \\ N & 0 \end{pmatrix}.$$

In $PG(2,q)$ any point lies on $q+1$ lines and any two points lie on exactly one common line. It follows that

$$N^2 = (q+1)I + (J-I).$$

Thus the eigenvalues of N^2 are $(q+1)^2$ and q with multiplicities 1 and q^2+q, and the eigenvalues of N are $q+1$, $\sqrt{q}$ and $-\sqrt{q}$ with multiplicities 1, m_1 and m_2. The values of m_1 and m_2 depend on a, but they are both non-zero. Thus we can apply Corollary 3.3 to bound the size of an independent set S in X:

$$|S| \leqslant \frac{\sqrt{q} + \sqrt{q + 4a\frac{q+\sqrt{q}+1}{q^2+q+1}}}{2\frac{q+\sqrt{q}+1}{q^2+q+1}}.$$

A similar approach yields bounds on the size of an independent set in polarities of the generalized quadrangles $W(q)$. This consists of a set of points, a set of lines, and an incidence relation between them such that

(a) Each point is incident with $q+1$ lines and two distinct points are incident with at most one common line.
(b) Each line is incident with $q+1$ points and two distinct lines are incident with at most one common point.
(c) Given a point p and a line l not incident with p, there exists a unique point q and a unique line m such that m is incident with p and q and q is incident with m and l.

The reader is directed to [14] for more details. The number of points in $W(q)$ is q^3+q^2+q+1, which is also the number of lines. Thus a polarity graph of $W(q)$ has q^3+q^2+q+1 vertices and degree $q+1$ (leaving the loops in). Computing the eigenvalues of a polarity graph of $W(q)$ is a little more work than for a polarity graph of $PG(2,q)$, but the argument is similar (see [14, Section 1.8.2] for details). The least eigenvalue is $-\sqrt{2q}$. Furthermore, the number of absolute points is always q^2+1. Thus Corollary 3.3 gives the following bound on the size of an independent set S in any polarity graph of $W(q)$:

$$|S| \leqslant \frac{\sqrt{2q}+\sqrt{2q+4(q^2+1)\frac{q+\sqrt{2q}+1}{q^3+q^2+q+1}}}{2\frac{q+\sqrt{2q}+1}{q^3+q^2+q+1}}.$$

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